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Self-Duality Equations on S^6 from $\mathbb{R}^7$ monopole

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Abstract

In this note we identify a correspondence between a seven-dimensional monopole configuration of the Yang-Mills-Higgs system and the generalized self-dual configuration of the Yang-Mills system on a six-dimensional sphere. In particular, the topological charge of the self-duality configurations belongs to the sixth homotopy group of the coset G/H associated with the symmetry breaking $G \to H$ induced by a non-trivial Higgs configuration in seven-dimensions.

In this short note we make an observation about the self-duality equations on the six-dimensional sphere. We make use of the work of [1, 2, 3, 4], the details of which we omit. It is well known [5] that a four-dimensional instanton configuration has second Chern character, which is in turn, related to the third homotopy group $\pi_3(G)$ of the gauge group G. We show there is a correspondence between seven-dimensional monopoles and self-duality equations on the six-dimensional sphere. There have been numerous efforts to generalize monopoles to higher dimensions, some of which have appeared in [1, 6, 7, 8, 9, 10].

In analogy in six-dimensions, when $G = SU(N)$, the third Chern character $\mathrm{Tr}F^3$ is considered as a topological charge and takes values in $\pi_5(G)$, with $\pi_5(SU(N)) = \mathbb{Z}$ for $N \geq 3$. In particular, for $SU(4) \simeq SO(6)$[1] pure Yang-Mills theory on S^6, one has a non-trivial gauge configuration [4], which satisfies the generalized self-duality relation

$$cF \wedge F = *_6 F. \tag{1}$$

Here, $c = 3/(\boldsymbol{q}R_0^2)$ is a covariantly constant scalar given in terms of the gauge coupling $\boldsymbol{q}$ and radius of S^6 R_0.

A few examples of other configurations for $\pi_5(G) \neq 0$ have appeared in the literature in [3]. In this note, our focus is non-trivial solutions of self-duality equations on S^6 with gauge group G with $\pi_5(G) = 0$.

In one dimension higher, the above equation takes the form

$$F \wedge F = *_7 \tilde{c}\{D\phi, F\}, \tag{2}$$

where $\tilde{c}$ is a constant. The above equation can be obtained from the Bogomol'nyi equation [9]. Here F is a gauge field strength two-form and "$*_7$" is the Hodge dual operator with respect to the Euclidean metric on $\mathbb{R}^7$. ϕ^a are scalar fields forming a fundamental multiplet of SO(7), $\phi := \phi^a \gamma_a$ and finally, D is the covariant exterior derivative: $D\phi = d\phi + g[A, \phi]$. The Hermitian matrices γ_a, $(a = 1, 2, \cdots, 7)$, are Dirac matrices with respect to SO(7), with $\gamma_{ab} := (1/2)[\gamma_a, \gamma_b]$ satisfying the commutation relations of SO(7). ϕ induces symmetry breaking when it acquires an expectation value $\|\langle\phi^a\rangle\| = H_0$.

To substantiate this connection, we suppose that the gauge configuration is concentrated around the origin of $\mathbb{R}^7$. Solutions of Eq. (2) represent monopole configurations with

[1] This is easily embedded in $SU(N)$ with $N \geq 4$.

corresponding topological charge,

$$Q = \int_{B(R_0)} \mathrm{Tr} D\phi F^3 = \int_{S^6_{R_0}} \mathrm{Tr} \phi F^3 \ , \tag{3}$$

where $B(R_0) = \{x \in \boldsymbol{R}^7 | \|x\| \leq R_0\}$. This charge Q relates to the mapping class degree of $S^6_{R_0} \to SO(7)/SO(6) = S^6$ for the case where $R_0 >> 1$. To see this, we suppose that gauge field A and scalar field ϕ have the following form,

$$A = \frac{1 - K(r)}{2\boldsymbol{q}} ede \ , \qquad \phi = H_0 U(r) e \ , \qquad e = \frac{x^a}{r} \gamma_a \ , \tag{4}$$

where $\boldsymbol{q}$ is again the gauge coupling, $r = \sqrt{x_a x^a}$ and the functions $U(r)$ and $K(r)$ satisfy the following boundary conditions: $U(0) = 1, K(0) = 1, U(\infty) = 1$ and $K(\infty) = 0$. The corresponding F and $D\phi$ are

$$F = \frac{1 - K^2}{4\boldsymbol{q}} de \wedge de - \frac{K'}{2\boldsymbol{q}} edr \wedge de \ , \qquad D\phi = H_0 (KUde + U'edr) \ . \tag{5}$$

For this particular configuration, Eq. (2) reduces to a first order nonlinear ordinary differential equation [9].

In the asymptotic region, F and $D\phi$ become

$$F \to \frac{1}{4\boldsymbol{q}} de \wedge de \ , \qquad D\phi \to H_0 U' edr \ , \tag{6}$$

where, as may be seen, F is aligned perpendicular to the radial direction and thus, along the S^6. Hence F can be regarded as a differential form on S^6. In this asymptotic region, Eq. (2) is tranformed into Eq. (1) with a suitable scalar.

However, the above discussion includes some degree of approximation: the self-duality is not exact. If we now relax the constraint of demanding a finite energy configuration by considering the singular configuration

$$A = \frac{1}{2\boldsymbol{q}} ede \ , \qquad \phi = -\frac{\kappa}{r} e \ , \tag{7}$$

where κ is a constant, the seven-dimensional equation

$$F \wedge F = *\mathbf{i}\mu \{D\phi, F\} \ , \qquad \mu = \frac{3}{2\boldsymbol{q}\kappa} \ , \tag{8}$$

reduces to Eq. (1).

Having constructed a concrete example, we now consider other embeddings. In general, we may consider a gauge group G with non-trivial $\pi_6(G/H)$ with symmetry breaking $G \to H$,

from a seven-dimensional monopole solution. For simplicity suppose that G is a simple group and the rank of group G is greater than or equal to 3. From the long exact sequence of homotopy group we obtain

$$\pi_6(G/H) \simeq \mathrm{Ker}\{\pi_5(H) \to \pi_5(G)\} \ . \tag{9}$$

If $\pi_5(G) = 0$ and H includes Spin(6) or SU(N) ($N \geq 3$) as a factor group, then $\pi_6(G/H) \neq 0$.

In contrast to the earlier example where the Higgs is in the fundamental **7** of $SO(7)$, it is possible to embed it and the adjoint **21** of $SO(7)$ in the adjoint **28** of $SO(8)$. Here $\pi_6(G/H) \neq 0$, and we can embed the above solution into the larger gauge theory with adjoint Higgs field and it does not come loose as a result of a gauge transformation of the larger group. E_8, SU(N), ($N \geq 8$) and SO(N) ($N \geq 8$) also permit the same configuration with adjoint Higgs. It would be interesting to explore embeddings of this configuration in string theory or M-theory: the gauge groups $SO(16)$ and E_8, both appear in [11]. For example, it may be possible to consider symmetry breakings $SO(16) \to SO(6) \times SU(5) \times U(1)$ and $E_8 \to SU(4) \times SU(5) \times U(1)$, inspired by the symmetry breaking of $SO(10)$ GUT: $SO(10) \to SU(5) \times U(1)$.

For these symmetry breakings $\pi_6(G/H) \neq 0$. It may also be of interest to consider coupling this system to gravity in a similar fashion to studies appearing in [12, 13, 14], the latter of which addresses the possibility of cosmological models as a result of dynamical compactification on S^6.

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